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MATHEMATICS

On the Chromatic Numbers of Some Distance Graphs

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Abstract—New estimates are obtained for the chromatic numbers of graphs from various classes of distance graphs with vertices in $\{-1,0,1\}^n$.

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In the context of the classical Hadwiger–Nelson problem concerning the chromatic number of a space (see [1]), Raigorodskii began to study the chromatic numbers of distance graphs with vertices in $\{-1,0,1\}^n$ (see [2]). Specifically, the following problem was posed in [3, 4].

Let n be a positive integer and l_{-1}, l_0, l_1 be positive integers summing to n. Additionally, let $b \in \mathbb{N}$. Define

$$V_n(l_{-1},l_0,l_1) = \{\mathbf{x} = (x_1,\dots,x_n): x_i \in \{-1,0,1\},$$
$$|\{i: x_i = -1\}| = l_{-1},$$
$$|\{i: x_i = 0\}| = l_0, \quad |\{i: x_i = 1\}| = l_1\},$$

$$E_n(l_{-1},l_0,l_1,b) = \{\{\mathbf{x},\mathbf{y}\}: |\mathbf{x}-\mathbf{y}| = b\},$$

$$G_n(l_{-1},l_0,l_1,b) = (V_n(l_{-1},l_0,l_1), E_n(l_{-1},l_0,l_1,b)).$$

The task is to find or estimate the quantity

$$\chi_1 = \chi(\{-1,0,1\}^n; l_{-1},l_0,l_1) = \max_b \chi(G_n(l_{-1},l_0,l_1,b)),$$

where $\chi(G)$ is the usual chromatic number of the graph G.

Two main theorems were proved in [3, 4]. Before formulating them, we introduce some notation. First, for $i_1, i_2 \in \{-1,0,1\}$, let $l(i_1,i_2) = l_{i_1} + l_{i_2}$,

$$V(l_{i_1},l_{i_2}) = \{\mathbf{x} = (x_1,\dots,x_{l(i_1,i_2)}): x_i \in \{i_1,i_2\},$$
$$|\{i: x_i = i_1\}| = l_{i_1}, |\{i: x_i = i_2\}| = l_{i_2}\}$$

$$\overline{s}_{i_1,i_2} = \max_{\mathbf{x},\mathbf{y}\in V(l_{i_1},l_{i_2})} (\mathbf{x},\mathbf{y}), \quad \underline{s}_{i_1,i_2} = \min_{\mathbf{x},\mathbf{y}\in V(l_{i_1},l_{i_2})} (\mathbf{x},\mathbf{y}),$$

$$\overline{s}_{-1,0,1} = \max_{\mathbf{x},\mathbf{y}\in V_n(l_{-1},l_0,l_1)} (\mathbf{x},\mathbf{y}), \quad \underline{s}_{-1,0,1} = \min_{\mathbf{x},\mathbf{y}\in V_n(l_{-1},l_0,l_1)} (\mathbf{x},\mathbf{y}).$$

Second, let $p_{0,1}$, $p_{-1,0}$, $p_{-1,1}$, and $p^m_{-1,0,1}$ be minimum odd primes ($m \geq 2$ is a positive integer that is not related to the dimension) such that

$$\overline{s}_{0,1} - 2p_{0,1} < \underline{s}_{0,1}, \quad \overline{s}_{-1,0} - 2p_{-1,0} < \underline{s}_{-1,0},$$

$$\overline{s}_{-1,1} - 8p_{-1,1} < \underline{s}_{-1,1}, \quad \overline{s}_{-1,0,1} - mp^m_{-1,0,1} < \underline{s}_{-1,0,1}.$$

Finally, let

$$P(l_{-1},l_0,l_1) = |V_n(l_{-1},l_0,l_1)| = \frac{n!}{l_{-1}!\,l_0!\,l_1!}.$$

In this notation, the following results are true.

Theorem A. *Let* $i_1 < i_2; i_1, i_2 \in \{-1,0,1\}$; *and* $i_3 \in \{-1, 0, 1\}$ *be the number remaining in the set* $\{-1,0,1\}$ *after* i_1, i_2 *have been removed from it. Define*

$$D_{i_1,i_2} = C_n^{l_{i_3}} \sum_{k_1=0}^{p_{i_1,i_2}-1} C^{k_1}_{l_{i_1}+l_{i_2}}.$$

Then

$$\chi_1 \geq \max_{i_1,i_2} \frac{P(l_{-1},l_0,l_1)}{D_{i_1,i_2}}.$$

Theorem B. *Let* $m \geq 2$ *be a positive integer independent of the dimension. Define*

$$D^m_{-1,0,1} = \sum_{(i,j)\in\mathcal{A}} C_n^i C_{n-i}^j,$$

where

$$\mathcal{A} = \{(i,j): i + j \leq n, i + 2j \leq p^m_{-1,0,1} - 1, i,j \in \mathbb{N} \cup \{0\}\}.$$

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Then

$$\chi_1 \geq \max_m \left(\frac{P(l_{-1}, l_0, l_1)}{D^m_{-1,0,1}} \right)^{\frac{1}{m-1}}.$$

FORMULATION OF THE RESULT

We managed to strengthen both results due to considerably wider choice of parameters $p_{0,1}$, $p_{-1,0}$, $p_{-1,1}$, and $p^m_{-1,0,1}$. It became fundamentally possible due to recent works [5–7]. The following new results are valid.

Theorem 1. *Let $i_1 < i_2; i_1, i_2 \in \{-1, 0, 1\}$; and $i_3 \in \{-1, 0, 1\}$ be the number remaining in the set $\{-1, 0, 1\}$ after i_1, i_2 have been removed from it. If, for the chosen i_1, i_2, we have one of the inequalities*

$$\overline{s}_{0,1} - 2p_{0,1} < \underline{s}_{0,1},$$

$$\overline{s}_{-1,0} - 2p_{-1,0} < \underline{s}_{-1,0}, \quad \overline{s}_{-1,1} - 8p_{-1,1} < \underline{s}_{-1,1},$$

where p_{i_1,i_2} is an odd prime, then we define

$$D_{i_1,i_2} = C_n^{l_{i_3}} \sum_{i=0}^{p_{i_1,i_2}-1} C^i_{l_{i_1}+l_{i_2}}.$$

If $i_3 \neq 0$ and, on the contrary, for the chosen i_1, i_2, we have one of the inequalities

$$\overline{s}_{0,1} - 2p_{0,1} \geq \underline{s}_{0,1}, \quad \overline{s}_{-1,0} - 2p_{-1,0} \geq \underline{s}_{-1,0},$$

where p_{i_1,i_2} is an odd prime, then t is defined as $t = \overline{s}_{i_1,i_2} - p_{i_1,i_2}$, we set $d = 2t - \overline{s}_{i_1,i_2} + 1$ and $k = \overline{s}_{i_1,i_2}$, consider $d_1, d_2 \in \mathbb{N} \cup \{0\}: d_1 + d_2 = d$, set $n_1 = n - l_{i_3} - d_1$ and $k_1 = \overline{s}_{i_1,i_2} - d_1$, define $r \in \mathbb{N}$ by the relation

$$(k_1 - d_2 + 1)\left(2 + \frac{d_2 - 1}{r+1}\right) \leq n_1 < (k_1 - d_2 + 1)\left(2 + \frac{d_2 - 1}{r}\right),$$

and set

$$D_{i_1,i_2} = C_n^{l_{i_3}} \frac{C_{n_1}^{d_2+2r} C^{d_1}_{n-l_{i_3}}}{C_{k_1}^{d_2+r} C^r_{n_1-k_1} C^{d_1}_k} \sum_{i=0}^{p_{i_1,i_2}-1} C^i_{n_1}.$$

Finally,

$$\chi_1 \geq \max_{i_1,i_2} \frac{P(l_{-1}, l_0, l_1)}{D_{i_1,i_2}}.$$

Theorem 2. *Let $m \geq 2$ be a positive integer independent of the dimension. If $\overline{s}_{-1,0,1} - mp^m_{-1,0,1} < \underline{s}_{-1,0,1}$ holds with some odd prime $p^m_{-1,0,1}$, then define*

$$D^m_{-1,0,1} = \sum_{(i,j)\in\mathcal{A}} C^i_n C^j_{n-i},$$

where

$$\mathcal{A} = \{(i,j): i + j \leq n, i + 2j \leq p^m_{-1,0,1} - 1, i, j \in \mathbb{N} \cup \{0\}\}.$$

If $\overline{s}_{-1,0,1} - mp^m_{-1,0,1} \geq \underline{s}_{-1,0,1}$, then define $d = \overline{s}_{-1,0,1} - mp^m_{-1,0,1} + 1$. Let $\mathfrak{m} = (m_1, m_2, m_3)$,

$$\mathfrak{M} = \begin{pmatrix} m_{-1,1} & m_{0,1} & m_{1,1} \\ m_{-1,2} & m_{0,2} & m_{1,2} \\ m_{-1,3} & m_{0,3} & m_{1,3} \end{pmatrix},$$

and all elements of the vector and the matrix are positive integers such that

$$m_1 + m_2 + m_3 = n,$$

$$m_{-1,1} + m_{0,1} + m_{1,1} = m_1,$$

$$m_{-1,2} + m_{0,2} + m_{1,2} = m_2, \quad m_{-1,3} + m_{0,3} + m_{1,3} = m_3,$$

$$m_{-1,1} + m_{-1,2} + m_{-1,3} = l_{-1},$$

$$m_{0,1} + m_{0,2} + m_{0,3} = l_0, \quad m_{1,1} + m_{1,2} + m_{1,3} = l_1$$

and another condition be satisfied. Suppose that the set $\{1, 2, \ldots, n\}$ is represented as a union of three disjoint parts M_1, M_2, M_3 of respective cardinalities m_1, m_2, m_3. A vector $\mathbf{x} = (x_1, \ldots, x_n) \in V_n(l_{-1}, l_0, l_1)$ is said to satisfy the partition $T(M_1, M_2, M_3)$ if

$$|\{i \in M_1 : x_i = -1\}| = m_{-1,1},$$

$$|\{i \in M_1 : x_i = 0\}| = m_{0,1}, \quad |\{i \in M_1 : x_i = 1\}| = m_{1,1},$$

$$|\{i \in M_2 : x_i = -1\}| = m_{-1,2},$$

$$|\{i \in M_2 : x_i = 0\}| = m_{0,2}, \quad |\{i \in M_2 : x_i = 1\}| = m_{1,2},$$

$$|\{i \in M_3 : x_i = -1\}| = m_{-1,3},$$

$$|\{i \in M_3 : x_i = 0\}| = m_{0,3}, \quad |\{i \in M_3 : x_i = 1\}| = m_{1,3}.$$

Another condition on the parameters $\mathfrak{m}$ and $\mathfrak{M}$ is that any two vectors $\mathbf{x}, \mathbf{y}$ satisfying the same partition have a scalar product equal to at least d. Define

$$D^m_{-1,0,1} = \frac{n!}{m_1! m_2! m_3!} \cdot \frac{m_{-1,1}! m_{-1,2}! m_{-1,3}!}{l_{-1}!} \cdot \frac{m_{0,1}! m_{0,2}! m_{0,3}!}{l_0!} \cdot \frac{m_{1,1}! m_{1,2}! m_{1,3}!}{l_1!} \cdot \sum_{(i,j)\in\mathcal{A}} C^i_n C^j_{n-i},$$

where

$$\mathcal{A} = \{(i,j): i + j \leq n, i + 2j \leq p^m_{-1,0,1} - 1\}.$$

Finally,

$$\chi_1 \geq \max_m \left(\frac{P(l_{-1}, l_0, l_1)}{D^m_{-1,0,1}} \right)^{\frac{1}{m-1}}.$$

Note that the improvements given by Theorems 1 and 2 as compared with Theorems A and B are the most significant in the cases $l_i \sim l'_l n$, $l'_l \in (0,1)$. In these cases, all estimates have the form $(c + o(1))^n$; moreover, for a large class of parameters l'_0, l'_1, l'_{-1}, the new values c are strictly greater than the previous ones, i.e., the well-known inequalities are strengthened exponentially.

Note also that similar problems for other parameter asymptotics were studied in recent works [8–15].

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